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Solution of a Problem of P. Turán on Zeros of Orthogonal Polynomials on the Unit Circle

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The existence of sequences of orthogonal polynomials on the unit circle whose zeros are everywhere dense in $|z| \leqslant 1$ is proved. © 1988 Academic Press, Inc.

P. Turán asked in [5, p. 68–69] whether there is a system of orthogonal polynomials on the unit circle such that the zeros of the polynomials are everywhere dense in $|z| \leqslant 1$. Szabados in [4, Theorem 7, p. 209] gave a partial answer. He proved that, given an arbitrary $\varepsilon > 0$, there exists a weight-function f such that the two-dimensional Lebesgue measure of the set of cluster points of the zeros of the orthonormal polynomials with respect to f is greater than $\pi - \varepsilon$.

The problem raised by Turán admits an affirmative answer as we show. We denote $D = \{z \in C: |z| < 1\}$; $\bar{D}$ is the closure of D, and T is the boundary of D.

LEMMA 1. *Let $\{\phi_h(z)\}_{h=0}^{n-1}$ be a finite sequence of orthogonal monic polynomials on T and $\alpha \in C$ with $|\alpha| < 1$. Then there exists only one monic polynomial $\phi_n(z)$ such that $\{\phi_h(z)\}_{h=0}^{n}$ is orthogonal on T and $\phi_n(\alpha) = 0$.*

Proof. Let φ_n^* be defined by $\varphi_n^*(z) = z^n \bar{\varphi}_n(1/z)$. Because $\phi_n(z)$ must satisfy the recurrence formula

$$\phi_n(z) = z\phi_{n-1}(z) - \overline{a_{n-1}}\,\phi_{n-1}^*(z), \qquad |a_{n-1}| < 1,$$

([2, Formula 8.1, p. 155]) we need only determine a_{n-1}. Since $\phi_n(\alpha) = 0$, we have

$$\alpha\phi_{n-1}(\alpha) = \overline{a_{n-1}}\,\phi_{n-1}^*(\alpha),$$

and thus, the recurrence coefficient is obtained.

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From the Christoffel–Darboux formula ([2, Formula 1.7, p. 8) we deduce

$$\frac{|\phi_n^*(z)|^2 - |\phi_n(z)|^2}{1-|z|^2} > 0$$

and so, therefore

$$|z| \gtreqless 1 \Rightarrow |\phi_n^*(z)| \lesseqgtr |\phi_n(z)|.$$

Because $|\alpha| < 1$, it follows that

$$|a_{n-1}| = |\alpha| \cdot \left| \frac{\phi_{n-1}(\alpha)}{\phi_{n-1}^*(\alpha)} \right| < 1,$$

which guarantees that a_{n-1} is the coefficient we were looking for.

One can see another proof of this result in [1].

Lemma 1 implies:

LEMMA 2. *Let* $\{\alpha_n\}_{n=1}^{\infty}$ *be a sequence of complex with* $|\alpha_n| < 1$, $n = 1, 2, \ldots$. *Then there exists only one family* $\{\phi_n(z)\}_{n=0}^{\infty}$ *of monic orthogonal polynomials on T such that* $\phi_0(z) = 1$ *and* $\phi_n(\alpha_n) = 0$, $n \geqslant 1$.

Lemma 2 assures that if we choose $\{\alpha_n\}_{n=1}^{\infty}$ dense in a closed subset F of D, the closure of the zeros of $\{\phi_n(z)\}_{n=0}^{\infty}$ contains F.

Because there is only one finite positive Borel measure associated with every sequence of orthogonal polynomials $\{\phi_n(z)\}_{n=0}^{\infty}$ ([2, Theorem 8.1, p. 156]) we obtain:

PROPOSITION. *There exist measures* $d\mu$ *on T such that the zeros of the orthogonal polynomials associated with* $d\mu$ *are everywhere dense in D.*

Finally, we want to point out that Lemma 2 assures that orthogonal polynomials on the unit circle are completely determined by some of their zeros. In [3], a similar result for orthogonal polynomials on the real line is proved. There, a connection between orthogonal polynomials, their zeros, and their recurrence coefficients is, besides, revealed.

One can also see another proof of the above result of Nevai–Totik in [6].

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